DETERMINING THE TIME DEPENDENT MATRIX POTENTIAL IN A WAVE EQUATION FROM PARTIAL BOUNDARY DATA

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Abstract. We study the inverse problem for determining the time-dependent matrix potential appearing in the wave equation. We prove the unique determination of potential from the knowledge of solution measured on a part of the boundary.

Keywords : Inverse problems, wave equation, Carleman estimates, partial boundary data, time-dependent coefficient

Mathematics Subject Classifications (2010): 35L05, 35L20, 35R30

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ be a bounded open set with C^2 boundary $\partial\Omega$. For $T > 0$, let $Q := (0,T) \times \Omega$ and we denote its lateral boundary by $\Sigma := (0,T) \times \partial\Omega$. Throughout this article, $\mathbf{H}^s(X)$ will denote the space of vector valued functions defined on X with each of its component belongs to $H^s(X)$. Similar notations will be used for other vector valued function spaces as well such as $\mathbf{C}^k(X)$, $\mathbf{L}^2(X)$ etc. Let $q(t,x) := (q_{ij}(t,x))_{1\leq i,j\leq n}$, is a time-dependent matrix valued potential with each $q_{ij} \in W^{1,\infty}(Q)$ and we write this as $q \in W^{1,\infty}(Q)$. For a displacement vector $\vec{u}(t,x) := (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^T$ and a matrix valued potential $q(t,x)$, we denote by $\mathcal{L}_q$ the following operator

$$\mathcal{L}_q\vec{u}(t,x) := \begin{bmatrix} \Box u_1(t,x) + \sum_{j=1}^n q_{1j}(t,x)u_j(t,x) \\ \Box u_2(t,x) + \sum_{j=1}^n q_{2j}(t,x)u_j(t,x) \\ \vdots \\ \Box u_n(t,x) + \sum_{j=1}^n q_{nj}(t,x)u_j(t,x) \end{bmatrix}, \quad (t,x) \in Q \tag{1}$$

where $\Box := \partial_t^2 - \Delta_x$, denotes the wave operator. Now we consider the following initial boundary value problem:

$$\begin{cases} \mathcal{L}_q\vec{u}(t,x) = \vec{0}, \ (t,x) \in Q \\ \vec{u}(0,x) = \vec{\phi}, \ \partial_t\vec{u}(0,x) = \vec{\psi}(x), \ x \in \Omega \\ \vec{u}(t,x) = \vec{f}(t,x), \ (t,x) \in \Sigma. \end{cases} \tag{2}$$

Using Theorem 2.1 in §2, if for $q \in L^\infty(Q)$, $\vec{\phi} \in \mathbf{H}^1(Q)$, $\vec{\psi} \in \mathbf{L}^2(\Omega)$ and $\vec{f} \in \mathbf{H}^1(\Sigma)$ is such that $\vec{f}(0,x) = \vec{\phi}(x)$ for $x \in \partial\Omega$, then there exists a unique solution $\vec{u}$ of (2) satisfying the following

$$\vec{u} \in \mathbf{C}^1\left([0,T];\mathbf{L}^2(\Omega)\right) \cap \mathbf{C}\left([0,T];\mathbf{H}^1(\Omega)\right) \text{ and } \partial_\nu\vec{u} \in \mathbf{L}^2(\Sigma),$$

where $\partial_\nu\vec{u}$ represents the component-wise normal derivative of vector $\vec{u}$, that is $\partial_\nu\vec{u} := (\partial_\nu u_1, \cdots, \partial_\nu u_n)$.

Based on this we define the continuous linear input-output operator $\Lambda_q : \mathbf{H}^1(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}^1(\Sigma) \to \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Sigma)$ by

$$\Lambda_q\left(\vec{\phi}, \vec{\psi}, \vec{f}\right) := \left(\vec{u}(T,\cdot), \partial_\nu \vec{u}|_\Sigma\right). \tag{3}$$

In this paper we consider the inverse problem of determining time-dependent potential q from the knowledge of input-output operator Λ_q measured on a subset of ∂Q. Our goal is to prove a uniqueness result for determining q from the partial information of Λ_q measured on ∂Q (see Theorem 3.1 below in §3 for more details).

Uniqueness issues for determining the coefficients in hyperbolic inverse problems are of great interest in last few decades. There have been extensive works in the literature regarding the identification of coefficients from boundary measurements involving the single wave equation while concerning the coefficients identification problems for the system of hyperbolic equations, not many results are available in literature. To the best of our knowledge the problem of determining the time-independent matrix potential appearing in a one dimensional wave equation from boundary measurement is first studied in [2] and recently this result has been extended in [18] to the determination of matrix valued potential using finite number of boundary measurements. Following the ideas used in [8], authors in [2] showed that the time-independent matrix potential can be recovered from the boundary measurements. Eskin and Ralston in [11] studied the problem of determining the first order as well as zeroth order time independent matrix valued perturbations in hyperbolic equations and proved the uniqueness up to a gauge invariance (see [12]) from the full boundary measurements. The gauge invariance appears only because of first (or higher) order perturbations and hence in the present work there will be no gauge invariance since we are only considering the zeroth order perturbation. Hence one can hope to recover the matrix potential q uniquely for the above system of equation (2) from the boundary measurements and this is the question we study in the current article. Next we mention the works related to the single wave equation which are closely related to the problem we study in this article. Unique determination for time independent scalar potential from boundary data appearing in (2) is initially studied by Bukhgeĭm and Klibanov in [9] (see also [33]). In [33] uniqueness was proved using the geometric optics solutions inspired by the work of Sylvester and Uhlmann [39] for elliptic problem. Rakesh and Ramm in [35] considered the unique determination of time-dependent scalar potential and they proved that the potential can be determined uniquely in some subset of Q from the knowledge of the Dirichlet to Neumann map measured on Σ. In [35] the wave equation with time-dependent potential in $\mathbb{R}\times\Omega$ is considered and they proved the uniqueness result for determining the coefficient from the Dirichlet to Neumann map measured on $\mathbb{R}\times\partial\Omega$. For finite time domain Q the problem for determining the time-dependent potential was studied by [22] where uniqueness result was proved using informations of the solutions at initial and final time in addition to the Dirichlet to Neumann map. Recently Kian in [26] proved that the uniqueness considered in [22] can be shown using the less information than that of [22]. Using the Carleman estimate together with geometric optic solutions Kian in [26] established the uniqueness for scalar time dependent potential using the informations of solution measured on a suitable subset of ∂Q. For anisotropic wave equation the unique determination for the time-dependent scalar potential from partial boundary data has been considered in [28]. For more works related to the determination of coefficients appearing in the single wave equation from boundary measurements, we refer to [1, 3, 4, 5, 6, 13, 24, 25, 26, 37, 38] and references therein.

In this paper we consider the unique determination of time-dependent matrix valued potential $q(t,x)$ appearing in (2) from the partial boundary data. Our work can be seen as an extension of the work of [26] who considered the aforementioned problem for determining the scalar time-dependent potential q appearing in (2).

The paper is organized as follows. In §2 we prove the well-posedness of the forward problem for Equation (2). In §3, we state the main result of the article. §4 is devoted to derive the Carleman estimates which will be used to prove the existence of geometric optics (GO) solutions and in §5, we construct the required GO solutions. Finally in §6, we prove the main theorem 3.1 of the article.

2. Preliminary result

In this section we prove the existence and uniqueness for the initial boundary value problem. In particular we prove the following theorem:

Theorem 2.1. *Let $q \in W^{1,\infty}(Q)$ be a time-dependent matrix potential. Suppose $\vec{\phi} \in \boldsymbol{H}^1(\Omega)$, $\vec{\psi} \in \boldsymbol{L}^2(\Omega)$ and $\vec{f} \in \boldsymbol{H}^1(\Sigma)$ is such that $\vec{f}(0,x) = \vec{\phi}(x)$ for $x \in \partial\Omega$. Then there exists a unique solution $\vec{u}$ to (2) satisfying the following*

$$\vec{u} \in \boldsymbol{C}^1\left([0,T];\boldsymbol{L}^2(\Omega)\right) \cap \boldsymbol{C}\left([0,T];\boldsymbol{H}^1(\Omega)\right) \text{ and } \partial_\nu \vec{u} \in \boldsymbol{L}^2(\Sigma).$$

Moreover, there exists a constant $C > 0$ depending only on q, T and Ω such that

$$\|\partial_\nu \vec{u}\|_{\boldsymbol{L}^2(\Sigma)} + \|\vec{u}\|_{\boldsymbol{H}^1(Q)} \leq C\left(\|\vec{\phi}\|_{\boldsymbol{H}^1(\Omega)} + \|\vec{\psi}\|_{\boldsymbol{L}^2(\Omega)} + \|\vec{f}\|_{\boldsymbol{L}^2(\Sigma)}\right) \tag{4}$$

holds.

Proof. Let us write the solution $\vec{u}$ to (2) into two terms as $\vec{u}(t,x) := \vec{v}(t,x) + \vec{w}(t,x)$ where $\vec{v}$ is solution to

$$\begin{cases} \partial_t^2 \vec{v}(t,x) - \Delta_x \vec{v}(t,x) = \vec{0}, \ (t,x) \in Q \\ \vec{v}(0,x) = \vec{\phi}(x), \ \partial_t \vec{v}(0,x) = \vec{\psi}(x), \ x \in \Omega \\ \vec{v}(t,x) = \vec{f}(t,x), \ (t,x) \in \Sigma \end{cases} \tag{5}$$

and $\vec{w}$ is solution to

$$\begin{cases} \mathcal{L}_q \vec{w}(t,x) = -q(t,x)\vec{v}(t,x), \ (t,x) \in Q \\ \vec{w}(0,x) = \partial_t \vec{w}(0,x) = \vec{0}, \ x \in \Omega \\ \vec{w}(t,x) = \vec{0}, \ (t,x) \in \Sigma. \end{cases} \tag{6}$$

Since Equation (5) is a decoupled system of wave equations therefore following Theorem 2.30 in [24] there exists a unique solution $\vec{v}(t,x)$ to (5) such that

$$\begin{aligned} &\vec{v} \in \mathbf{C}^1\left([0,T];\mathbf{L}^2(\Omega)\right) \cap \mathbf{C}\left([0,T];\mathbf{H}^1(\Omega)\right) \text{ and } \partial_\nu \vec{v} \in \mathbf{L}^2(\Sigma) \\ &\text{and } \|\partial_\nu \vec{v}\|_{\mathbf{L}^2(\Sigma)} + \|\vec{v}\|_{\mathbf{H}^1(Q)} \leq C\left(\|\vec{\phi}\|_{\mathbf{H}^1(\Omega)} + \|\vec{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\vec{f}\|_{\mathbf{L}^2(\Sigma)}\right) \end{aligned} \tag{7}$$

holds for some constant $C > 0$ independent of $\vec{v}$. Using Equation (7) and the fact that $q \in W^{1,\infty}(Q)$, we have $q\vec{v} \in \mathbf{L}^2(Q)$. Now following the arguments from [24, 30, 32] we prove the existence and uniqueness for $\vec{w}$ solution to (6). We define the time-dependent bilinear form $a(t;\cdot,\cdot)$ on $\mathbf{H}_0^1(\Omega)$ by

$$a(t;\vec{h},\vec{g}) := \int_\Omega \left(\nabla_x \vec{h}(x) \cdot \overline{\nabla_x \vec{g}(x)} + q(t,x)\vec{h}(x) \cdot \overline{\vec{h}(x)}\right) \mathrm{d}x, \text{ for } \vec{h}, \vec{g} \in \mathbf{H}_0^1(\Omega). \tag{8}$$

Since $\vec{h}, \vec{g}$ are time-independent and $q \in L^{\infty}(Q)$ therefore for each fixed $\vec{h}, \vec{g} \in \mathbf{H}_0^1(\Omega)$ we have $a(t;\vec{h},\vec{g}) \in L^{\infty}(0,T)$. Also using the Cauchy-Schwartz inequality and the fact that $q \in L^{\infty}(Q)$ we get

$$|a(t;\vec{h},\vec{g})| \leq C\|\vec{h}\|_{\mathbf{H}_0^1(\Omega)}\|\vec{g}\|_{\mathbf{H}_0^1(\Omega)} \tag{9}$$

where constant $C>0$ is independent of $\vec{h}$ and $\vec{g}$. Next consider

$$\begin{aligned}|a(t;\vec{h},\vec{h})| &= \Big|\int\limits_{\Omega}\left(|\nabla_x\vec{h}(x)|^2 + q(t,x)\vec{h}(x)\cdot\overline{\vec{h}(x)}\right)\mathrm{d}x\Big|\\ &\geq \|\nabla_x\vec{h}\|^2_{\mathbf{L}^2(\Omega)} - \|q\|_{L^{\infty}(Q)}\|\vec{h}\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

Choosing $\lambda > \|q\|_{L^{\infty}(Q)}$ in above equation, we get

$$|a(t;\vec{h},\vec{h})| + \lambda\|\vec{h}\|^2_{\mathbf{L}^2(\Omega)} \geq \alpha\|\vec{h}\|^2_{\mathbf{H}^1(\Omega)}, \text{ for some constant } \alpha>0. \tag{10}$$

Combining Equations (8), (9) and (10), we get that $t \mapsto a(t;\vec{h},\vec{g})$ is continuous bilinear form for all $\vec{h},\vec{g} \in \mathbf{H}_0^1(\Omega)$. Also note that the principle part of $a(t;\cdot,\cdot)$ given by

$$a(t;\vec{h},\vec{g}) = \int\limits_{\Omega}\nabla_x\vec{h}(x)\cdot\overline{\nabla_x\vec{g}(x)}\mathrm{d}x \tag{11}$$

is anti-symmetric. Therefore using Theorem 8.1 together with Remark 8.1 of Chapter 3 in [30] (see also [32]), we have that the initial boundary value problem given by (6) admits a unique solution $\vec{w} \in \mathbf{C}^1\left([0,T];\mathbf{L}^2(\Omega)\right) \cap \mathbf{C}\left([0,T];\mathbf{H}^1(\Omega)\right)$ and it satisfies the following estimate

$$\int\limits_{Q}\left(|\vec{w}(t,x)|^2 + |\partial_t\vec{w}(t,x)|^2 + |\nabla_x\vec{w}(t,x)|^2\right)\mathrm{d}x\mathrm{d}t \leq C\left(\|\vec{\phi}\|_{\mathbf{H}^1(\Omega)} + \|\vec{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\vec{f}\|_{\mathbf{L}^2(\Sigma)}\right). \tag{12}$$

Next we prove that $\partial_\nu\vec{w} \in \mathbf{L}^2(\Sigma)$. We follow the arguments similar to the one used in [31] for the wave equation with scalar potential. Let $\nu(x)$ denote the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. We extend this to $\overline{\Omega}$ and denote the extended one by $\nu(x)$ itself. Now consider the following integral

$$\begin{aligned}\int\limits_{Q}(T-t)\,\mathcal{L}_q\vec{w}(t,x)\cdot(\nu(x)\cdot\nabla_x\vec{w}(t,x))\,\mathrm{d}x\mathrm{d}t &= \int\limits_{Q}(T-t)\,\partial_t^2\vec{w}(t,x)\cdot(\nu(x)\cdot\nabla_x\vec{w}(t,x))\,\mathrm{d}x\mathrm{d}t\\ &-\int\limits_{Q}(T-t)\,\Delta_x\vec{w}(t,x)\cdot(\nu(x)\cdot\nabla_x\vec{w}(t,x))\,\mathrm{d}x\mathrm{d}t + \int\limits_{Q}(T-t)\,q(t,x)\vec{w}(t,x)\cdot(\nu(x)\cdot\nabla_x\vec{w}(t,x))\,\mathrm{d}x\mathrm{d}t\\ &= \sum_{j=1}^{n}\int\limits_{Q}(T-t)\,\partial_t^2 w_j(t,x)\,(\nu(x)\cdot\nabla_x w_j(t,x))\,\mathrm{d}x\mathrm{d}t - \sum_{j=1}^{n}\int\limits_{Q}(T-t)\,\Delta_x w_j(t,x)\,(\nu(x)\cdot\nabla_x w_j(t,x))\,\mathrm{d}x\mathrm{d}t\\ &+\sum_{i,j=1}^{n}\int\limits_{Q}(T-t)\,q_{ij}(t,x)w_j(t,x)\,(\nu(x)\cdot\nabla_x w_j(t,x))\,\mathrm{d}x\mathrm{d}t := A_1 + A_2 + A_3\end{aligned}$$

where

$$
\begin{aligned}
A_1 &:= \sum_{j=1}^{n} \int\limits_{Q} (T-t)\, \partial_t^2 w_j(t,x)\, (\nu(x)\cdot \nabla_x w_j(t,x))\, \mathrm{d}x\mathrm{d}t \\
A_2 &:= -\sum_{j=1}^{n} \int\limits_{Q} (T-t)\, \Delta_x w_j(t,x)\, (\nu(x)\cdot \nabla_x w_j(t,x))\, \mathrm{d}x\mathrm{d}t \\
A_3 &:= \sum_{i,j=1}^{n} \int\limits_{Q} (T-t)\, q_{ij}(t,x) w_j(t,x)\, (\nu(x)\cdot \nabla_x w_i(t,x))\, \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Using Equation (6), we have

$$
A_1 + A_2 + A_3 = -\int\limits_{Q} (T-t) q(t,x) \vec{v}(t,x) \cdot (\nu(x)\cdot \nabla_x \vec{w}(t,x))\, \mathrm{d}x\mathrm{d}t. \tag{13}
$$

We simplify each of A_j for $1 \leq j \leq 3$. Using integration parts, we have A_1 is

$$
\begin{aligned}
A_1 &= -T \sum_{j=1}^{n} \int\limits_{\Omega} \partial_t w_j(0,x)\, (\nu(x)\cdot \nabla_x w_j(0,x))\, \mathrm{d}x + \sum_{j=1}^{n} \int\limits_{Q} \partial_t w_j(t,x)\, (\nu(x)\cdot \nabla_x w_j(t,x))\, \mathrm{d}x\mathrm{d}t \\
&\quad - \sum_{j=1}^{n} \int\limits_{Q} (T-t)\, \partial_t w_j(t,x)\, (\nu(x)\cdot \nabla_x \partial_t w_j(t,x))\, \mathrm{d}x\mathrm{d}t \\
&= -T \int\limits_{\Omega} \partial_t \vec{w}(0,x) \cdot (\nu(x)\cdot \nabla_x \vec{w}(0,x))\, \mathrm{d}x + \int\limits_{Q} \partial_t \vec{w}(t,x) \cdot (\nu(x)\cdot \nabla_x \vec{w}(t,x))\, \mathrm{d}x\mathrm{d}t \\
&\quad - \int\limits_{Q} \frac{T-t}{2} \nabla_x \cdot \left(\nu(x) |\partial_t \vec{w}(t,x)|^2\right) \mathrm{d}x\mathrm{d}t + \int\limits_{Q} \frac{T-t}{2} |\partial_t \vec{w}(t,x)|^2 \nabla_x \cdot \nu(x) \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Using the Gauss divergence theorem and the fact that $\vec{w}|_{\Sigma} = \vec{w}|_{t=0} = \partial_t \vec{w}|_{t=0} = 0$, we get

$$
A_1 = \int\limits_{Q} \partial_t \vec{w}(t,x) \cdot (\nu(x)\cdot \nabla_x \vec{w}(t,x))\, \mathrm{d}x\mathrm{d}t + \int\limits_{Q} \frac{T-t}{2} |\partial_t \vec{w}(t,x)|^2 \nabla_x \cdot \nu(x) \mathrm{d}x\mathrm{d}t. \tag{14}
$$

Now using the integration by parts in the expression for A_2, we have

$$
\begin{aligned}
A_2 &= -\sum_{j=1}^{n}\int_Q (T-t)\,\Delta_x w_j(t,x)\,(\nu(x)\cdot\nabla_x w_j(t,x))\,\mathrm{d}x\mathrm{d}t\\
&= -\sum_{j=1}^{n}\int_Q (T-t)\sum_{k,l=1}^{n}\partial_k^2 w_j(t,x)\nu_l(x)\partial_l w_j(t,x)\mathrm{d}x\mathrm{d}t\\
&= -\sum_{j=1}^{n}\int_Q (T-t)\nabla_x\cdot(\nabla_x w_j(t,x)\nu(x)\cdot\nabla_x w_j(t,x))\,\mathrm{d}x\mathrm{d}t - \int_Q \frac{T-t}{2}\nabla_x\cdot\nu(x)|\nabla_x\vec{w}(t,x)|^2\mathrm{d}x\mathrm{d}t\\
&\quad + \sum_{j=1}^{n}\int_Q (T-t)\sum_{k,l=1}^{n}\partial_k w_j(t,x)\partial_k\nu_l(x)\partial_l w_j(t,x)\mathrm{d}x\mathrm{d}t + \int_Q \frac{T-t}{2}\nabla_x\cdot\left(\nu(x)|\nabla_x\vec{w}(t,x)|^2\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Gauss divergence theorem and $\vec{u}|_\Sigma = 0$, gives

$$
\begin{aligned}
A_2 = &-\int_\Sigma \frac{T-t}{2}|\partial_\nu\vec{w}(t,x)|^2\mathrm{d}S_x\mathrm{d}t + \sum_{j=1}^{n}\int_Q (T-t)\sum_{k,l=1}^{n}\partial_k w_j(t,x)\partial_k\nu_l(x)\partial_l w_j(t,x)\mathrm{d}x\mathrm{d}t\\
&-\int_Q \frac{T-t}{2}\nabla_x\cdot\nu(x)|\nabla_x\vec{w}(t,x)|^2\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{15}
$$

Finally using Equations (14), (15) and the Cauchy-Schwartz inequality in (13), we get

$$
\Big|\int_\Sigma \frac{T-t}{2}|\partial_\nu\vec{w}(t,x)|^2\mathrm{d}S_x\mathrm{d}t\Big| \leq C\int_Q \left(|\vec{v}(t,x)|^2 + |\vec{w}(t,x)|^2 + |\partial_t\vec{w}(t,x)|^2 + |\nabla_x\vec{w}(t,x)|^2\right)\mathrm{d}x\mathrm{d}t.
$$

Hence using (7) and (12) in the above equation, we get

$$
\Big|\int_\Sigma \frac{T-t}{2}|\partial_\nu\vec{w}(t,x)|^2\mathrm{d}S_x\mathrm{d}t\Big| \leq C\left(\|\vec{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\vec{f}\|_{\mathbf{L}^2(\Sigma)}\right).
$$

Thus, we have shown the following

$$
\begin{aligned}
&\vec{w}\in\mathbf{C}^1\left([0,T];\mathbf{L}^2(\Omega)\right)\cap\mathbf{C}\left([0,T];\mathbf{H}^1(\Omega)\right) \text{ and } \partial_\nu\vec{w}\in\mathbf{L}^2(\Sigma)\\
&\text{and } \|\partial_\nu\vec{w}\|_{\mathbf{L}^2(\Sigma)} + \|\vec{w}\|_{\mathbf{H}^1(Q)} \leq C\left(\|\vec{\phi}\|_{\mathbf{H}^1(\Omega)} + \|\vec{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\vec{f}\|_{\mathbf{L}^2(\Sigma)}\right).
\end{aligned}
\tag{16}
$$

Now combining Equations (7) and (16), we get

$$
\begin{aligned}
&\vec{u}\in\mathbf{C}^1\left([0,T];\mathbf{L}^2(\Omega)\right)\cap\mathbf{C}\left([0,T];\mathbf{H}^1(\Omega)\right) \text{ and } \partial_\nu\vec{u}\in\mathbf{L}^2(\Sigma)\\
&\text{and } \|\partial_\nu\vec{u}\|_{\mathbf{L}^2(\Sigma)} + \|\vec{u}\|_{\mathbf{H}^1(Q)} \leq C\left(\|\vec{\phi}\|_{\mathbf{H}^1(\Omega)} + \|\vec{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\vec{f}\|_{\mathbf{L}^2(\Sigma)}\right).
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

3. Statement of the main result

Before stating the main result of this article, we introduce some notation. Following [10], for fix $\omega_0\in\mathbb{S}^{n-1}$ and define

$$
\partial\Omega_{+,\omega_0} := \{x\in\partial\Omega:\ \nu(x)\cdot\omega_0\geq 0\},\quad \partial\Omega_{-,\omega_0} := \{x\in\partial\Omega:\ \nu(x)\cdot\omega_0\leq 0\}
$$

where $\nu(x)$ is outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Corresponding to $\partial\Omega_{\pm,\omega_0}$, we denote the lateral boundary parts by $\Sigma_{\pm,\omega_0} := (0,T) \times \partial\Omega_{\pm,\omega_0}$. We denote by $F = (0,T) \times F'$ and $G = (0,T) \times G'$ where F' and G' are small enough open neighbourhoods of $\partial\Omega_{+,\omega_0}$ and $\partial\Omega_{-,\omega_0}$ respectively in $\partial\Omega$. Now let $\vec{u}$ be the solution to Equation (2) with $\vec{\phi} \in \mathbf{H}^1(\Omega)$, $\vec{\psi} \in \mathbf{L}^2(\Omega)$ and $\vec{f} \in \mathbf{H}^1(\Sigma)$ such that $\vec{f}(0,x) = \vec{\phi}(x)$ for $x \in \partial\Omega$. Next using Theorem 2.1, we can define our continuous linear input-output operator $\widetilde{\Lambda_q} : \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Sigma) \to \mathbf{H}^1(\Omega) \times \mathbf{L}^2(G)$ given by

$$\widetilde{\Lambda_q}(\vec{\phi}, \vec{\psi}, \vec{f}) = \left(\vec{u}|_{t=T}, \partial_\nu \vec{u}|_G\right) \tag{17}$$

where $\vec{u}$ is the solution to (2). In this paper, our aim is to prove the following uniqueness result for determining q from the knowledge of $\widetilde{\Lambda}_q$.

Theorem 3.1. *Let $q^{(1)}(t,x)$ and $q^{(2)}(t,x)$ be two sets of potentials such that the components of each $q^{(i)}$ are in $W^{1,\infty}(Q)$ for $i = 1,2$. Let $\vec{u}^{(i)}$ be solutions to* (2) *when $q = q^{(i)}$ and $\widetilde{\Lambda}_{q^{(i)}}$ for $i = 1,2$ be the input-output operators defined by* (3) *corresponding to $\vec{u}^{(i)}$. If*

$$\widetilde{\Lambda}_{q^{(1)}}(\vec{\phi}, \vec{\psi}, \vec{f}) = \widetilde{\Lambda}_{q^{(2)}}(\vec{\phi}, \vec{\psi}, \vec{f}), \ \text{for } (\vec{\phi}, \vec{\psi}, \vec{f}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^1(\Sigma), \tag{18}$$

then

$$q^{(1)}(t,x) = q^{(2)}(t,x), \ (t,x) \in Q.$$

To the best of our knowledge the problem considered here has not been studied and infact this is the first result which deals with the determination of time-dependent matrix valued coefficients appearing in hyperbolic partial differential equations from the boundary measurements. Theorem 3.1, can be proved by using the Carleman estimate together with constructing the geometric optics solutions for the wave equation with matrix valued potential. For time dependent scalar potential case this approach for hyperbolic inverse problems first appeared in [25, 26] and recently this approach has been used in [7, 19, 27, 28, 29] for determining the coefficients in the single wave equations. To prove Theorem 3.1, we follow the arguments similar to [7, 25, 26].

4. Carleman Estimate

The present section is devoted to deriving a Carleman estimate for (2) involving the boundary terms and it will be used to control the boundary terms over subsets of the boundary where measurements are not available. In order to state the Carleman estimate, first we will fix some notation. For $\vec{v} = (v_1, v_2, v_3, \cdots, v_n)^T \in \mathbf{H}^1(Q)$, we define the L^2 norm of $\vec{v}$ by

$$\|\vec{v}\|_{\mathbf{L}^2(Q)} := \left(\sum_{j=1}^{n} \int_Q |v_j(t,x)|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} = \left(\sum_{j=1}^{n} \|v_j\|^2_{L^2(Q)}\right)^{1/2}$$

and

$$\nabla_x \vec{v} := (\nabla_x v_1, \nabla_x v_2, \nabla_x v_3, \cdots, \nabla_x v_n)^T \ \text{and } \omega \cdot \nabla_x \vec{v} := (\omega \cdot \nabla_x v_1, \omega \cdot \nabla_x v_2, \cdots, \omega \cdot \nabla_x v_n)^T.$$

Theorem 4.1. *Let $\varphi(t,x) := t + x\cdot\omega$, where $\omega \in \mathbb{S}^{n-1}$ is fixed and $q \in L^\infty(Q)$. Then the Carleman estimate*

$$\begin{aligned}
&\|e^{-\varphi/h}\vec{u}\|^2_{\boldsymbol{L}^2(Q)} + h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu\vec{u}, e^{-\varphi/h}\partial_\nu\vec{u}\right)_{\boldsymbol{L}^2(\Sigma_{+,\omega})} + h\left(e^{-\varphi(T,\cdot)/h}\partial_t\vec{u}(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t\vec{u}(T,\cdot)\right)_{\boldsymbol{L}^2(\Omega)}\\
&\leq C\Bigg(\|he^{-\varphi/h}\mathcal{L}_q\vec{u}\|^2_{\boldsymbol{L}^2(Q)} + \frac{1}{h}\left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}\vec{u}(T,\cdot)\right)_{\boldsymbol{L}^2(\Omega)}\\
&\qquad + h\left(e^{-\varphi(T,\cdot)/h}\nabla_x\vec{u}(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x\vec{u}(T,\cdot)\right)_{\boldsymbol{L}^2(\Omega)} + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu\vec{u}, e^{-\varphi/h}\partial_\nu\vec{u}\right)_{\boldsymbol{L}^2(\Sigma_{-,\omega})}\Bigg)
\end{aligned}\tag{19}$$

holds for all $\vec{u} \in \boldsymbol{C}^{\,2}(Q)$ with

$$\vec{u}|_\Sigma = 0,\ \ \vec{u}|_{t=0} = \partial_t\vec{u}|_{t=0} = 0,$$

and h small enough.

Proof. Define the conjugated operator $\Box_\varphi$ by

$$\Box_\varphi := h^2 e^{-\varphi/h}\Box e^{\varphi/h}.\tag{20}$$

For $\vec{v} \in \mathbf{C}^2(Q)$, we have

$$\Box_\varphi\vec{v}(t,x) = h^2\Box\vec{v}(t,x) + 2h\left(\partial_t - \omega\cdot\nabla_x\right)\vec{v}(t,x) := P_1\vec{v}(t,x) + P_2\vec{v}(t,x)$$

where

$$P_1\vec{v}(t,x) = h^2\Box\vec{v}(t,x)\ \ \text{and}\ \ P_2\vec{v}(t,x) = 2h\left(\partial_t - \omega\cdot\nabla_x\right)\vec{v}(t,x).$$

Now $\mathbf{L}^2$ norm of $\Box_\varphi\vec{v}$ for $\vec{v} \in \mathbf{C}^2(Q)$ satisfying $\vec{v}|_\Sigma = \vec{v}|_{t=0} = \partial_t\vec{v}|_{t=0} = 0$, can be estimated as

$$\begin{aligned}
\int_Q |\Box_\varphi\vec{v}(t,x)|^2\mathrm{d}x\mathrm{d}t &= \int_Q |P_1\vec{v}(t,x)|^2\mathrm{d}x\mathrm{d}t + \int_Q |P_2\vec{v}(t,x)|^2\mathrm{d}x\mathrm{d}t + 2\int_Q \mathrm{Re}\left(P_1\vec{v}(t,x)\cdot\overline{P_2\vec{v}(t,x)}\right)\mathrm{d}x\mathrm{d}t\\
&\geq \int_Q |P_2\vec{v}(t,x)|^2\mathrm{d}x\mathrm{d}t + 2\int_Q \mathrm{Re}\left(P_1\vec{v}(t,x)\cdot\overline{P_2\vec{v}(t,x)}\right)\mathrm{d}x\mathrm{d}t\\
&= 4h^2\int_Q |\left(\partial_t - \omega\cdot\nabla_x\right)\vec{v}(t,x)|^2\mathrm{d}x\mathrm{d}t + 4h^3\int_Q \mathrm{Re}\left(\Box\vec{v}(t,x)\cdot\overline{\partial_t\vec{v}(t,x)}\right)\mathrm{d}x\mathrm{d}t\\
&\quad - 4h^3\int_Q \mathrm{Re}\left(\Box\vec{v}(t,x)\cdot\left(\omega\cdot\overline{\nabla_x\vec{v}(t,x)}\right)\right)\mathrm{d}x\mathrm{d}t\\
&= 4h^2\sum_{j=1}^n\int_Q |\left(\partial_t - \omega\cdot\nabla_x\right)v_j(t,x)|^2\mathrm{d}x\mathrm{d}t + 4h^3\sum_{j=1}^n\int_Q \mathrm{Re}\left(\Box v_j(t,x)\overline{\partial_t v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t\\
&\quad - 4h^3\sum_{j=1}^n\int_Q \mathrm{Re}\left(\Box v_j(t,x)\left(\omega\cdot\overline{\nabla_x v_j(t,x)}\right)\right)\mathrm{d}x\mathrm{d}t\\
&:= \sum_{j=1}^n\left(I_{1,j} + I_{2,j} + I_{3,j}\right),
\end{aligned}$$

where

$$\begin{aligned} I_{1,j} &:= 4h^2 \int_Q |(\partial_t - \omega\cdot\nabla_x)\, v_j(t,x)|^2 \mathrm{d}x\mathrm{d}t \\ I_{2,j} &:= 4h^3 \int_Q \mathrm{Re}\left(\Box v_j(t,x)\overline{\partial_t v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ I_{3,j} &:= -4h^3 \int_Q \mathrm{Re}\left(\Box v_j(t,x)\left(\omega\cdot\overline{\nabla_x v_j(t,x)}\right)\right)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{21}$$

We will estimate each of $I_{k,j}$ for $1 \le k \le 3$ and each fixed $1 \le j \le n$. We first simplify $I_{1,j}$. To estimate $I_{1,j}$, first consider the following integral for $0 \le s \le T$

$$2\int_0^s\int_\Omega \left(\partial_t v_j(t,x) - \omega\cdot\nabla_x v_j(t,x)\right) v_j(t,x)\mathrm{d}x\mathrm{d}t = \int_\Omega |v_j(s,x)|^2\mathrm{d}x - \int_0^s\int_\Omega \nabla_x\cdot\left(|v_j(t,x)|^2\omega\right)\mathrm{d}x\mathrm{d}t.$$

Now using Cauchy-Schwartz inequality on left hand side of the above equation and the fact that $v_j(t,x)|_\Sigma = 0$, we have

$$\int_\Omega |v_j(s,x)|^2\mathrm{d}x \le \frac{1}{\epsilon^2}\int_0^s\int_\Omega |(\partial_t - \omega\cdot\nabla_x)\, v_j(t,x)|^2\mathrm{d}x\mathrm{d}t + \epsilon^2\int_0^s\int_\Omega |v_j(t,x)|^2\mathrm{d}x\mathrm{d}t \tag{22}$$

holds for any $\epsilon > 0$. Now integrating both sides of (22) with respect to s variable from 0 to T, we have

$$\int_0^T\int_\Omega |v_j(s,x)|^2\mathrm{d}x\mathrm{d}s \le \frac{T}{\epsilon^2}\int_0^T\int_\Omega |(\partial_t - \omega\cdot\nabla_x)\, v_j(t,x)|^2\mathrm{d}x\mathrm{d}t + T\epsilon^2\int_0^T\int_\Omega |v_j(t,x)|^2\mathrm{d}x\mathrm{d}t.$$

Now choose $\epsilon > 0$, small enough such that $1 - T\epsilon^2 > 0$, we get

$$4Ch^2\int_0^T\int_\Omega |v_j(t,x)|^2\mathrm{d}x\mathrm{d}t \le I_{1,j} \tag{23}$$

where $C > 0$ is some constant depending only on T. Next using the integration by parts and the fact that $v_j|_\Sigma = v_j|_{t=0} = \partial_t v_j|_{t=0} = 0$, we have $I_{2,j}$ is

$$\begin{aligned} I_{2,j} &= 4h^3\int_Q \mathrm{Re}\left(\Box v_j(t,x)\overline{\partial_t v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &= 2h^3\int_Q \frac{\partial}{\partial t}|\partial_t v_j(t,x)|^2\mathrm{d}x\mathrm{d}t - 4h^3\int_Q \mathrm{Re}\left(\Delta v_j(t,x)\overline{\partial_t v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &= 2h^3\int_\Omega \left(|\partial_t v_j(T,x)|^2 + |\nabla_x v_j(T,x)|^2\right)\mathrm{d}x. \end{aligned}$$

Finally, we consider $I_{3,j}$. This is

$$I_{3,j} = -4h^3 \int\limits_Q \mathrm{Re}\left(\Box v_j(t,x)\omega \cdot \overline{\nabla_x v_j(t,x)}\right) \mathrm{d}x\mathrm{d}t.$$

We have

$$\begin{aligned}
I_{3,j} &= -4h^3 \mathrm{Re} \int\limits_Q \partial_t^2 v_j(t,x)\overline{\omega\cdot\nabla_x v_j(t,x)}\mathrm{d}x\mathrm{d}t + 4h^3 \mathrm{Re}\int\limits_Q \Delta v_j(t,x)\overline{\omega\cdot\nabla_x v_j(t,x)}\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int\limits_Q \partial_t\left(\partial_t v_j(t,x)\overline{\omega\cdot\nabla_x v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t + 4h^3\mathrm{Re}\int\limits_Q \partial_t v_j(t,x)\overline{\omega\cdot\nabla_x\partial_t v_j(t,x)}\mathrm{d}x\mathrm{d}t \\
&\quad + 4h^3\mathrm{Re}\int\limits_Q \nabla_x\cdot\left(\nabla_x v_j(t,x)\overline{\omega\cdot\nabla_x v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t - 4h^3\mathrm{Re}\int\limits_Q \nabla_x v_j(t,x)\cdot\nabla_x\left(\overline{\omega\cdot\nabla_x v_j(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int\limits_\Omega \partial_t v_j(T,x)\overline{\omega\cdot\nabla_x v_j(T,x)}\mathrm{d}x + 2h^3\int\limits_Q \nabla_x\cdot\left(\omega|\partial_t v_j(t,x)|^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad + 2h^3\mathrm{Re}\int\limits_\Sigma \partial_\nu v_j(t,x)\overline{\omega\cdot\nabla_x v_j(t,x)}\mathrm{d}S_x\mathrm{d}t - 2h^3\int\limits_Q \nabla_x\cdot\left(\omega|\nabla_x v_j|^2\right)\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int\limits_\Omega \partial_t v_j(T,x)\overline{\omega\cdot\nabla_x v_j(T,x)}\mathrm{d}x + 2h^3\int\limits_\Sigma \omega\cdot\nu|\partial_\nu v_j|^2\mathrm{d}S_x\mathrm{d}t.
\end{aligned}$$

In deriving the above equation, we used the fact that

$$2h^3\mathrm{Re}\int\limits_\Sigma \partial_\nu v_j(t,x)\overline{\omega\cdot\nabla_x v_j(t,x)}\mathrm{d}S_x\mathrm{d}t = 2h^3\int\limits_\Sigma \omega\cdot\nu|\partial_\nu v_j|^2\mathrm{d}S_x\mathrm{d}t,$$

since $v_j = 0$ on Σ. Also note that $\partial_t v_j(t,x) = 0$ and $|\nabla_x v_j| = |\partial_\nu v_j|$ on Σ.

Therefore

$$\begin{aligned}
\int\limits_Q |\Box_\varphi v_j(t,x)|^2\mathrm{d}x\mathrm{d}t \geq 4Ch^2\int\limits_0^T\int\limits_\Omega |v_j(t,x)|^2 + 2h^3\int\limits_\Omega\left(|\partial_t v_j(T,x)|^2 + |\nabla_x v_j(T,x)|^2\right)\mathrm{d}x \\
- 4h^3\mathrm{Re}\int\limits_\Omega \partial_t v_j(T,x)\overline{\omega\cdot\nabla_x v_j(T,x)}\mathrm{d}x + 2h^3\int\limits_\Sigma \omega\cdot\nu|\partial_\nu v_j|^2\mathrm{d}S_x\mathrm{d}t.
\end{aligned}$$

After using the Cauchy-Schwartz inequality to estimate third term, we get

$$\begin{aligned}
C\Big(h^2\int\limits_Q|\vec{v}(t,x)|^2 + h^3\int\limits_\Omega|\partial_t\vec{v}(T,x)|^2\mathrm{d}x - 4h^3\int\limits_\Omega|\nabla_x\vec{v}(T,x)|^2\mathrm{d}x \\
+2h^3\int\limits_\Sigma \omega\cdot\nu|\partial_\nu\vec{v}|^2\mathrm{d}S_x\mathrm{d}t\Big) \leq C\int\limits_Q|\Box_\varphi\vec{v}(t,x)|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{24}$$

Now we consider the conjugated operator $\mathcal{L}_\varphi := h^2 e^{-\frac{\varphi}{h}}\mathcal{L}_q e^{\frac{\varphi}{h}}$. We have

$$\mathcal{L}_\varphi\vec{v}(t,x) = h^2\left(e^{-\varphi/h}\left(\Box + q\right)e^{\varphi/h}\vec{v}(t,x)\right) = \Box_\varphi\vec{v}(t,x) + h^2 q(t,x)\vec{v}(t,x).$$

By triangle inequality,

$$\int_Q |\mathcal{L}_\varphi \vec{v}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \geq \frac{1}{2}\int_Q |\Box_\varphi \vec{v}(t,x)|^2 \mathrm{d}x\mathrm{d}t - h^4 \int_Q |q(t,x)\vec{v}(t,x)|^2 \mathrm{d}x\mathrm{d}t. \tag{25}$$

We have

$$h^4 \int_Q |q(t,x)\vec{v}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \leq Ch^4 \int_Q |\vec{v}(t,x)|^2 \mathrm{d}x\mathrm{d}t$$

where constant $C > 0$ depends on $\|q\|_{L^\infty(Q)}$. Using this together with Equation (24) in (25), we have that there exists a constant $C > 0$ depending only on T, Ω and q such that

$$\begin{aligned}C\left(h^2\int_Q |\vec{v}(t,x)|^2 + h^3\int_\Omega |\partial_t \vec{v}(T,x)|^2 \mathrm{d}x + 2h^3 \int_\Sigma \omega\cdot\nu |\partial_\nu \vec{v}|^2 \mathrm{d}S_x \mathrm{d}t\right)\\ \leq \int_Q |\mathcal{L}_\varphi \vec{v}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + 4h^3 \int_\Omega |\nabla_x \vec{v}(T,x)|^2 \mathrm{d}x\end{aligned}$$

and this inequality holds for h small enough. After dividing by h^2, we get

$$\begin{aligned}C\left(\int_Q |\vec{v}(t,x)|^2 + h\int_\Omega |\partial_t \vec{v}(T,x)|^2 \mathrm{d}x + 2h \int_\Sigma \omega\cdot\nu |\partial_\nu \vec{v}|^2 \mathrm{d}S_x \mathrm{d}t\right)\\ \leq \frac{1}{h^2}\int_Q |\mathcal{L}_\varphi \vec{v}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + 4h \int_\Omega |\nabla_x \vec{v}(T,x)|^2 \mathrm{d}x.\end{aligned} \tag{26}$$

Let us now substitute $\vec{v}(t,x) = e^{-\frac{\varphi}{h}}\vec{u}(t,x)$. We have

$$\begin{aligned}&he^{-\varphi/h}\partial_t u_j(t,x) = h\partial_t v_j + e^{-\varphi/h} u_j,\\ &he^{-\varphi/h}\nabla_x u_j = h\nabla_x v_j + e^{-\varphi/h}\omega u_j,\\ &\partial_\nu v_j(t,x)|_\Sigma = e^{-\varphi/h}\partial_\nu u_j|_\Sigma, \quad \text{since } u_j = 0 \text{ on } \Sigma.\end{aligned}$$

Using the triangle inequality, we have

$$h\int_\Omega e^{-2\varphi(T,x)/h}|\partial_t u_j(T,x)|^2 \mathrm{d}x - \frac{1}{h}\int_\Omega e^{-2\varphi(T,x)/h}|u_j(T,x)|^2\mathrm{d}x \leq Ch\int_\Omega e^{-2\varphi(T,x)/h}|\partial_t v_j(T,x)|^2 \mathrm{d}x$$

$$h\int_\Omega |\nabla_x v_j(T,x)|^2 \mathrm{d}x \leq C\left(h\int_\Omega e^{-2\varphi(T,x)/h}|\nabla_x u_j(T,x)|^2 \mathrm{d}x + \frac{1}{h}\int_\Omega e^{-2\varphi(T,x)/h}|u_j(T,x)|^2 \mathrm{d}x\right).$$

Using the above inequalities and choosing h small enough, we have

$$\int_Q e^{-2\varphi/h}|\vec{u}(t,x)|^2 \mathrm{d}x\mathrm{d}t + h\int_\Omega e^{-2\varphi(T,x)/h}|\partial_t u_j(T,x)|^2\mathrm{d}x + 2h\int_\Sigma \omega\cdot\nu(x) e^{-2\varphi/h}|\vec{u}(t,x)|^2 \mathrm{d}S_x \mathrm{d}t$$

$$\leq C\left(h^2\int_Q e^{-2\varphi/h}|\mathcal{L}_q \vec{u}(t,x)|^2 \mathrm{d}x\mathrm{d}t + h\int_\Omega e^{-2\varphi(T,x)/h}|\nabla_x \vec{u}(T,x)|^2 \mathrm{d}x + \frac{1}{h}\int_\Omega e^{-2\varphi(T,x)/h}|\vec{u}(T,x)|^2 \mathrm{d}x\right).$$

Finally,

$$\|e^{-\varphi/h}\vec{u}\|^2_{\mathbf{L}^2(Q)} + h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu\vec{u}, e^{-\phi/h}\partial_\nu\vec{u}\right)_{\mathbf{L}^2(\Sigma_{+,\omega})} + h\left(e^{-\varphi(T,\cdot)/h}\partial_t\vec{u}(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t\vec{u}(T,\cdot)\right)_{\mathbf{L}^2(\Omega)}$$
$$\leq C\Bigg(\|he^{-\varphi/h}\mathcal{L}_q\vec{u}\|^2_{\mathbf{L}^2(Q)} + \frac{1}{h}\left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}\vec{u}(T,\cdot)\right)_{\mathbf{L}^2(\Omega)}$$
$$+ h\left(e^{-\varphi(T,\cdot)/h}\nabla_x\vec{u}(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x\vec{u}(T,\cdot)\right)_{\mathbf{L}^2(\Omega)} + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu\vec{u}, e^{-\varphi/h}\partial_\nu\vec{u}\right)_{\mathbf{L}^2(\Sigma_{-,\omega})}\Bigg).$$

This completes the proof. □

5. Construction of Geometric optics solutions

Aim of this section is to construct exponential growing and decaying solutions which will be used to prove the main result of this article. To construct these solutions we follow very closely the ideas from [25, 26] used for constructing the geometric optics solutions for the wave equation with a scalar potential. We state the following lemma which will be used for constructing the solutions. Proof of this is given in [25].

Lemma 5.1. [26] *Let $\Box_{\pm\varphi}$ be as defined in* (20)*, then for each $0 < h < 1$ there exists a bounded linear operator $\Box^*_{\pm\varphi} : H^1(Q) \to H^1(Q)$ such that*

(1) $\Box^*_{\pm\varphi}\left(\Box_{\pm\varphi} f\right) = f,\ f \in H^1(Q)$
(2) $\|\Box^*_{\pm\varphi}\|_{\mathcal{B}(L^2(Q))} \leq C$
(3) $\Box^*_{\pm\varphi} \in \mathcal{B}\left(H^1(Q); H^2(Q)\right)$ and $\|\Box^*_{\pm\varphi}\|_{\mathcal{B}(H^1(Q);H^2(Q))} \leq C$

for some constant $C > 0$ depending only on Q.

Using Lemma 5.1 in the following Proposition, we construct the exponential decaying solution for $\mathcal{L}_{q^*}v = 0$.

Proposition 5.2. *Let q and φ be as in Theorem 4.1. Then, there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $\vec{v}_d \in \boldsymbol{H}^2(Q)$ satisfying $\mathcal{L}_{q^*}\vec{v}_d = 0$ of the form*

$$\vec{v}_d(t,x) = e^{-\frac{\varphi}{h}}\left(\vec{B}_d(t,x) + h\vec{R}_d(t,x;h)\right), \tag{27}$$

where

$$\vec{B}_d(t,x) = e^{-i\zeta\cdot(t,x)}\vec{K}_1 \tag{28}$$

with $\zeta \in (1,-\omega)^\perp$, $\vec{K}_1$ is a constant n-vector and $\vec{R}_d \in \boldsymbol{H}^2(Q)$ satisfies

$$\|\vec{R}_d\|_{\boldsymbol{L}^2(Q)} \leq C. \tag{29}$$

Proof. We have

$$\mathcal{L}_{q^*}\vec{v}(t,x) = \begin{bmatrix} \Box v_1(t,x) + \sum_{j=1}^n \overline{q}_{j1}(t,x)v_j(t,x) \\ \Box v_2(t,x) + \sum_{j=1}^n \overline{q}_{j2}(t,x)v_j(t,x) \\ \vdots \\ \Box v_n(t,x) + \sum_{j=1}^n \overline{q}_{jn}(t,x)v_j(t,x) \end{bmatrix}$$

and we are looking for $\vec{v}_d(t,x)$ of the form (27) such that

$$\mathcal{L}_{q^*}\vec{v}_d(t,x) = 0.$$

Thus, we have

$$\Box v_{di}(t,x) + \sum_{j=1}^{n} \overline{q}_{ji}(t,x) v_{dj}(t,x) = 0, \text{ for } 1 \leq i \leq n \tag{30}$$

where v_{di} stands for the ith component of $\vec{v}_d$. Also we denote by B_{di} and R_{di} as the ith component of $\vec{B}_d$ and $\vec{R}_d$ respectively. Now using the expressions for v_{di} from (27) in (30), we have

$$h^2 \Box R_{di} - 2h \left(\partial_t - \omega \cdot \nabla_x\right) R_{di} + h^2 \sum_{j=1}^{n} \overline{q}_{ji} R_{dj} = -h \Box B_{di} - h \sum_{j=1}^{n} \overline{q}_{ji} B_{dj}$$

holds for $1 \leq i \leq n$. Using Equation (20), we have

$$\Box_{-\varphi} \vec{R}_d(t,x) = -h \mathcal{L}_{q^*} \vec{B}_d(t,x) - h^2 q^*(t,x) \vec{R}_d(t,x). \tag{31}$$

Now for $\vec{w} \in \mathbf{H}^1(Q)$, we define the map $\mathcal{F} : \mathbf{H}^1(Q) \to \mathbf{H}^1(Q)$ by

$$\mathcal{F}(\vec{w}) := \Box_{-\varphi}^{*} \left(-h \mathcal{L}_{q^*} \vec{B}_d - h^2 q^* \vec{w} \right).$$

which is well-defined from Lemma 5.1 and the fact that $q \in W^{1,\infty}(Q)$. Now using Lemma 5.1, we have

$$\|\mathcal{F}(\vec{w}_1) - \mathcal{F}(\vec{w}_2)\|_{\mathbf{H}^1(Q)} = h^2 \left\| \Box_{-\varphi}^{*} \left(q^* \left\{ \vec{w}_1 - \vec{w}_2 \right\} \right) \right\|_{\mathbf{H}^1(Q)} \leq C h^2 \|\vec{w}_1 - \vec{w}_2\|_{\mathbf{H}^1(Q)}$$

for some constant $C > 0$ independent of $\vec{w}_i$ and h. Now choosing $h > 0$ small enough such that $Ch^2 < 1$, we have by fixed point theorem, there exists $\vec{w} \in \mathbf{H}^1(Q)$ such that $\mathcal{F}(\vec{w}) = \vec{w}$. Now going back to Equation (31) and using Lemma 5.1, we have $\vec{R}_d \in \mathbf{H}^2(Q)$ and $\|\vec{R}_d\|_{\mathbf{L}^2(Q)} \leq C$. This completes the proof of Proposition 5.2. $\square$

Next in the following proposition we construct the exponential growing solution to $\mathcal{L}_q \vec{v} = 0$.

Proposition 5.3. *Let q and φ be as in Theorem 4.1. Then, there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $\vec{v}_g \in \boldsymbol{H}^2(Q)$ satisfying $\mathcal{L}_q \vec{v}_g = 0$ of the form*

$$\vec{v}_g(t,x) = e^{\frac{\varphi}{h}} \left(\vec{B}_g(t,x) + h \vec{R}_g(t,x;h) \right), \tag{32}$$

$\vec{B}_g(t,x) := \vec{K}_2$ is a constant n-vector and $\vec{R}_g \in \boldsymbol{H}^2(Q)$ satisfies

$$\|\vec{R}_g\|_{\boldsymbol{L}^2(Q)} \leq C. \tag{33}$$

Proof. Proof follows by using the similar arguments as used in proving Proposition 5.2. $\square$

6. Recovery of q

In this section, we prove the main Theorem 3.1 of this article. The proof is based on deriving an integral identity followed by using the Carleman estimate and geometric optic solutions constructed in §5, we conclude the proof of our main result. To derive the integral identity, let us consider $\vec{u}^{(j)}$ be the solutions to the following initial boundary value problems with matrix valued potential $q^{(j)}$ for $j = 1, 2$.

$$\begin{cases} \mathcal{L}_{q^{(j)}} \vec{u}^{(j)}(t,x) = 0, \ (t,x) \in Q \\ \vec{u}^{(j)}(0,x) = \vec{\phi}(x), \ \partial_t \vec{u}^{(j)}(0,x) = \vec{\psi}(x), \ x \in \Omega \\ \vec{u}^{(j)}(t,x) = \vec{f}(t,x), \ (t,x) \in \Sigma. \end{cases} \tag{34}$$

Also denote

$$\begin{aligned}\vec{u}(t,x) &:= \vec{u}^{(1)}(t,x) - \vec{u}^{(2)}(t,x)\\ q(t,x) &:= q^{(2)}(t,x) - q^{(1)}(t,x).\end{aligned} \tag{35}$$

Then $\vec{u}$ will satisfies the following initial boundary value problem:

$$\begin{cases} \mathcal{L}_{q^{(1)}}\vec{u}(t,x) = q(t,x)\vec{u}^{(2)}(t,x),\ (t,x) \in Q\\ \vec{u}(0,x) = \partial_t \vec{u}(0,x) = \vec{0},\ x \in \Omega\\ \vec{u}(t,x) = \vec{0},\ (t,x) \in \Sigma \end{cases} \tag{36}$$

Let $\vec{v}(t,x)$ of the form given by (27) be the solution to following equation

$$\mathcal{L}^*_{q^{(1)}}\vec{v}(t,x) = 0 \text{ in } Q. \tag{37}$$

Also let $\vec{u}^{(2)}$ of the form given by (32) be solution to the following equation

$$\begin{cases} \mathcal{L}_{q^{(2)}}\vec{u}^{(2)}(t,x) = 0,\ (t,x) \in Q\\ \vec{u}^{(2)}(0,x) = \vec{\phi}(x),\ \partial_t \vec{u}^{(2)}(0,x) = \vec{\psi}(x),\ x \in \Omega\\ \vec{u}^{(2)}(t,x) = \vec{f}(t,x),\ (t,x) \in \Sigma. \end{cases} \tag{38}$$

Using Theorem 2.1, we have $\vec{u} \in \mathbf{H}^1(Q)$ and $\partial_\nu \vec{u} \in \mathbf{L}^2(\Sigma)$. Multiply (36) by $\overline{\vec{v}(t,x)} \in \mathbf{H}^1(Q)$ solution to (37) and integrate over Q. Now using integration by parts and taking into account the following: $\vec{u}|_\Sigma = \vec{0}$, $\vec{u}(T,x) = \vec{0}$, $\partial_\nu \vec{u}|_G = \vec{0}$, $\vec{u}|_{t=0} = \partial_t \vec{u}|_{t=0} = \vec{0}$ and $\mathcal{L}^*_{q^{(1)}}\vec{v}(t,x) = \vec{0}$, we get

$$\int_Q q(t,x)\vec{u}^{(2)}(t,x)\cdot\overline{\vec{v}(t,x)}\mathrm{d}x\mathrm{d}t = \int_\Omega \partial_t \vec{u}(T,x)\cdot\overline{\vec{v}(T,x)}\mathrm{d}x - \int_{\Sigma\setminus G} \partial_\nu \vec{u}(t,x)\cdot\overline{\vec{v}(t,x)}\mathrm{d}S_x\mathrm{d}t. \tag{39}$$

Lemma 6.1. *Let $\vec{u}^{(i)}$ for $i = 1,2$ solutions to (34) with $\vec{u}^{(2)}$ of the form (32). Let $\vec{u}(t,x) = \vec{u}^{(1)}(t,x) - \vec{u}^{(2)}(t,x)$, and $\vec{v}$ be of the form (27). Then*

$$h\int_\Omega \partial_t \vec{u}(T,x)\cdot\overline{\vec{v}(T,x)}\mathrm{d}x \to 0 \ \textit{as}\ h \to 0^+. \tag{40}$$

$$h\int_{\Sigma\setminus G} \partial_\nu \vec{u}(t,x)\cdot\overline{\vec{v}(t,x)}\mathrm{d}S_x\mathrm{d}t \to 0 \ \textit{as}\ h \to 0^+. \tag{41}$$

Proof. Using (27), (29) and Cauchy-Schwartz inequality, we get

$$\left|h\int_{\Omega}\partial_t\vec{u}(T,x)\cdot\overline{\vec{v}(T,x)}\mathrm{d}x\right| \leq \int_{\Omega} h\left|\partial_t\vec{u}(T,x)\cdot e^{-\frac{\varphi(T,x)}{h}}\overline{\left(\vec{B}_d(T,x)+h\vec{R}_d(T,x)\right)}\right|\mathrm{d}x$$

$$\leq C\left(\int_{\Omega} h^2\left|\partial_t\vec{u}(T,x)e^{-\frac{\varphi(T,x)}{h}}\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|e^{-i\xi\cdot(T,x)}\vec{K}_1+h\overline{\vec{R}_d(T,x)}\right|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_{\Omega} h^2\left|\partial_t\vec{u}(T,x)e^{-\frac{\varphi(T,x)}{h}}\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(1+\|h\vec{R}_d(T,\cdot)\|^2_{\mathbf{L}^2(\Omega)}\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_{\Omega} h^2\left|\partial_t\vec{u}(T,x)e^{-\frac{\varphi(T,x)}{h}}\right|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Now using the boundary Carleman estimate (4.1), we get,

$$h\int_{\Omega}\left|\partial_t\vec{u}(T,x)e^{-\frac{\varphi(T,x)}{h}}\right|^2\mathrm{d}x \leq C\|he^{-\varphi/h}\mathcal{L}_{q^{(1)}}\vec{u}\|^2_{\mathbf{L}^2(Q)} = C\|he^{-\varphi/h}q\vec{u}^{(2)}\|^2_{\mathbf{L}^2(Q)}.$$

Substituting (32) for $\vec{u}^{(2)}$, we get,

$$h\int_{\Omega}\partial_t\vec{u}(T,x)\cdot\overline{\vec{v}(T,x)}\mathrm{d}x \to 0 \text{ as } h\to 0^+.$$

For $\varepsilon>0$, define

$$\partial\Omega_{+,\varepsilon,\omega}=\{x\in\partial\Omega:\nu(x)\cdot\omega>\varepsilon\},\ \text{and}\ \Sigma_{+,\varepsilon,\omega}=(0,T)\times\partial\Omega_{+,\varepsilon,\omega}.$$

Next we prove (41). Since $\Sigma\setminus G\subseteq\Sigma_{+,\varepsilon,\omega}$ for all ω such that $|\omega-\omega_0|\leq\varepsilon$, substituting $\vec{v}=\vec{v}_d$ from (27) in (41) we have

$$\left|\int_{\Sigma\setminus G}\partial_\nu\vec{u}(t,x)\cdot\overline{\vec{v}(t,x)}\mathrm{d}S_x\mathrm{d}t\right| \leq \int_{\Sigma+,\varepsilon,\omega}\left|\partial_\nu\vec{u}(t,x)\cdot e^{-\frac{\varphi}{h}}\left(\vec{B}_d+h\vec{R}_d\right)(t,x)\right|\mathrm{d}S_x\mathrm{d}t$$

$$\leq C\left(1+\|h\vec{R}_d\|^2_{L^2(\Sigma)}\right)^{\frac{1}{2}}\left(\int_{\Sigma+,\varepsilon,\omega}\left|\partial_\nu\vec{u}(t,x)e^{-\frac{\varphi}{h}}\right|^2\mathrm{d}S_x\mathrm{d}t\right)$$

with $C>0$ is independent of h and this inequality holds for all ω such that $|\omega-\omega_0|\leq\varepsilon$. Now using trace theorem, we have that $\|\vec{R}_d\|_{\mathbf{L}^2(\Sigma)}\leq C\|\vec{R}_d\|_{\mathbf{H}^1(Q)}$. Using this, we get

$$\left|\int_{\Sigma\setminus G}\partial_\nu\vec{u}(t,x)\cdot\vec{v}(t,x)\mathrm{d}S_x\mathrm{d}t\right|\leq C\left(\int_{\Sigma+,\varepsilon,\omega}\left|\partial_\nu\vec{u}(t,x)e^{-\frac{\varphi}{h}}\right|^2\mathrm{d}S_x\mathrm{d}t\right)^{\frac{1}{2}}.$$

Now

$$\int_{\Sigma_{+,\varepsilon,\omega}} \left|\partial_\nu \vec{u}(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t = \frac{1}{\varepsilon}\int_{\Sigma_{+,\varepsilon,\omega}} \varepsilon \left|\partial_\nu \vec{u}(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t$$
$$\leq \frac{1}{\varepsilon}\int_{\Sigma_{+,\varepsilon,\omega}} \partial_\nu\varphi \left|\partial_\nu \vec{u}(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t.$$

Using (19), we have

$$\frac{h}{\varepsilon}\int_{\Sigma_{+,\varepsilon,\omega}} \partial_\nu\varphi \left|\partial_\nu \vec{u}(t,x)e^{-\frac{\varphi}{h}}\right|^2 \mathrm{d}S_x\mathrm{d}t \leq C\|he^{-\varphi/h}\mathcal{L}_{q^{(1)}}\vec{u}\|^2_{\mathbf{L}^2(Q)}.$$

Now proceeding as before, we get

$$h\int_{\Sigma\setminus G} \partial_\nu \vec{u}(t,x)\cdot\overline{\vec{v}(t,x)}\mathrm{d}S_x\mathrm{d}t \to 0 \text{ as } h\to 0^+.$$

$\square$

Substituting (32) for $\vec{u}^{(2)}$ and (27) for $\vec{v}$ in (39) and using (40) and (41), we get

$$\int_{\mathbb{R}^{1+n}} e^{-\mathrm{i}\xi\cdot(t,x)}q(t,x)\vec{K}_1\cdot\vec{K}_2\mathrm{d}x\mathrm{d}t = 0, \text{ for } \xi\in(1,-\omega)^\perp, \text{ for constant vectors } \vec{K}_1,\ \vec{K}_2 \text{ and } \omega \text{ near } \omega_0.$$

The set of all ξ such that $\xi\in(1,-\omega)^\perp$ for ω near ω_0 forms an open cone and since $q\in W^{1,\infty}(Q)$ has compact support therefore using the Paley-Wiener theorem we conclude that $q(t,x)\vec{K}_1\cdot\vec{K}_2 = 0$ for all $(t,x)\in Q$ and arbitrary constant vector $\vec{K}_1$ and $\vec{K}_2$. Thus, we have $q_1(t,x) = q_2(t,x)$. This completes the proof of Theorem 3.1.

Acknowledgments

The work of second author is supported by NSAF grant (No. U1930402).

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